Planar Graphs

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They/Them

dpheath.github.io
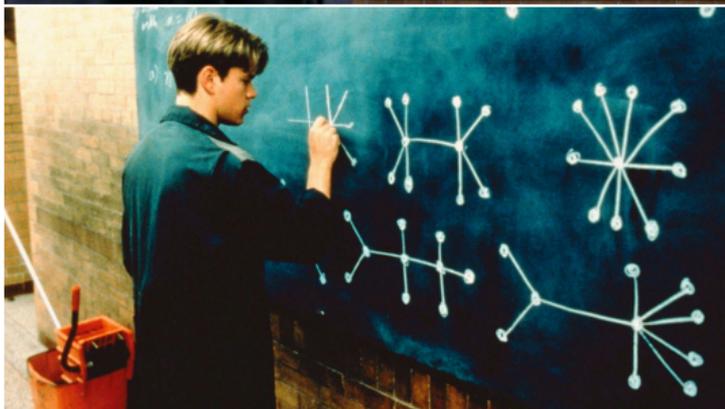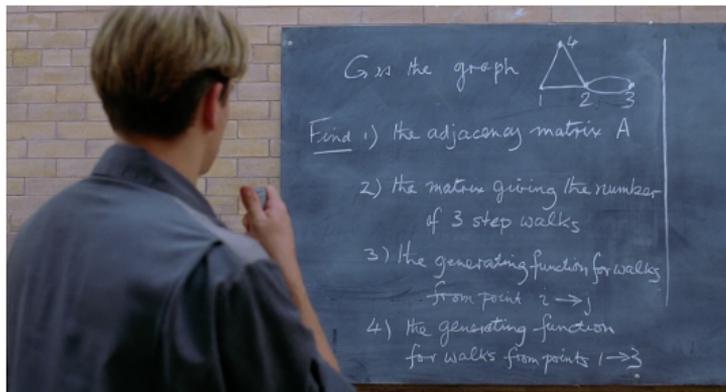
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Good Will Hunting

"Find all the homeographically irreducible trees on 10 vertices"
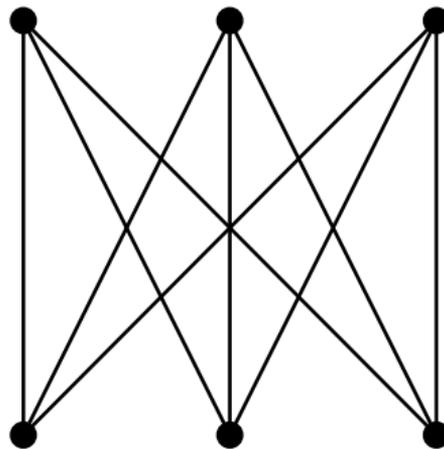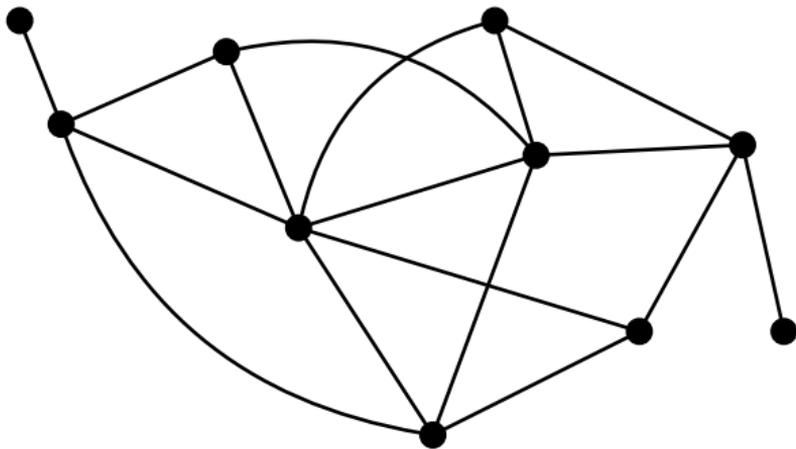
Utilities Problem

What is a graph?

A *graph* consists of two things:

- A collection of points called *vertices*...
- ... and a collection of connections called *edges* between them.
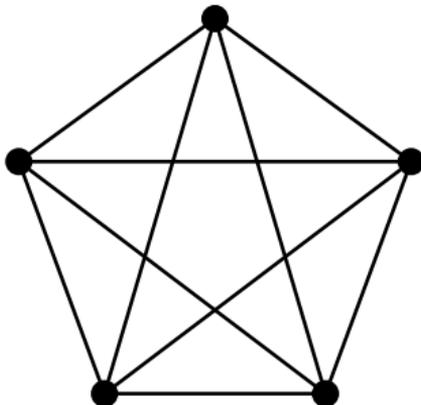
For example...

Planar Graphs

Question: Given a set of vertices and edges, can we draw the graph without crossing edges?

We call the graph *planar* if we can.

Fact: The Two Utilities Graph $\{AC, AD, BC, BD\}$ is planar.

Fact: The Three Utilities Graph $\{AD, AE, AF, BD, BE, BF, CD, CE, CF\}$ is **not** planar.
There is **no** way to draw it without crossing edges.

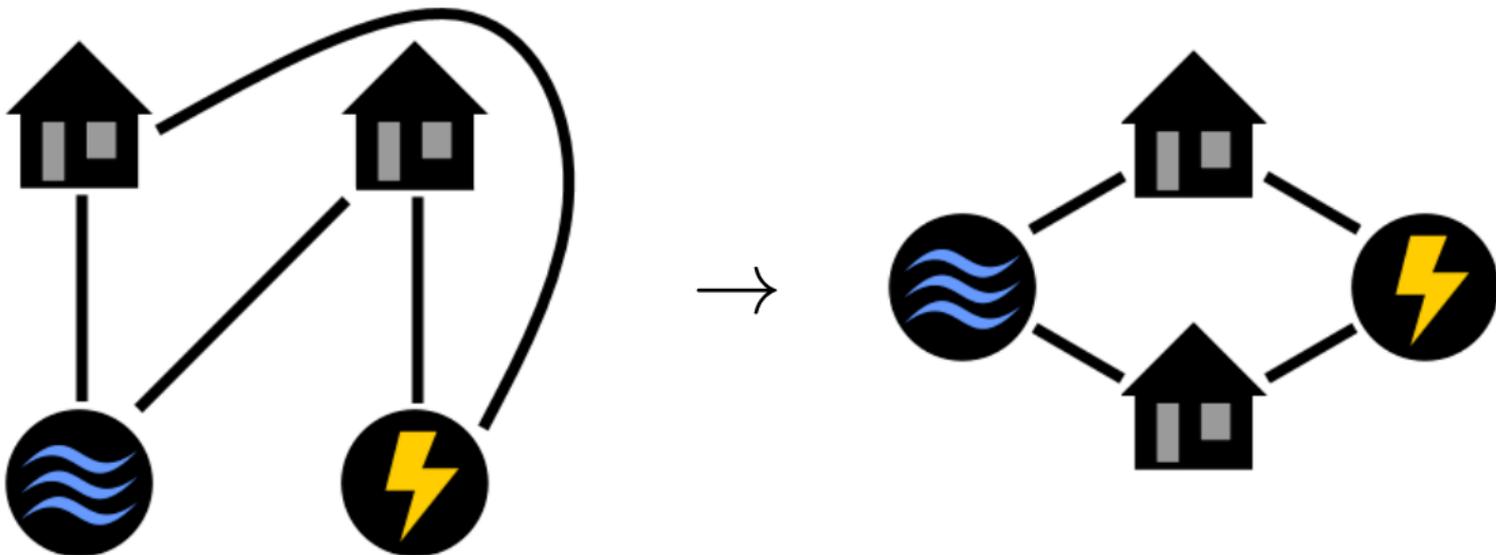**Fact:** The 'Pentagram' is **not** planar.

Can you think of any other graphs which are not planar?

More Questions

Question: If we have a graph that is planar, can we always move vertices around so that the edges are straight?

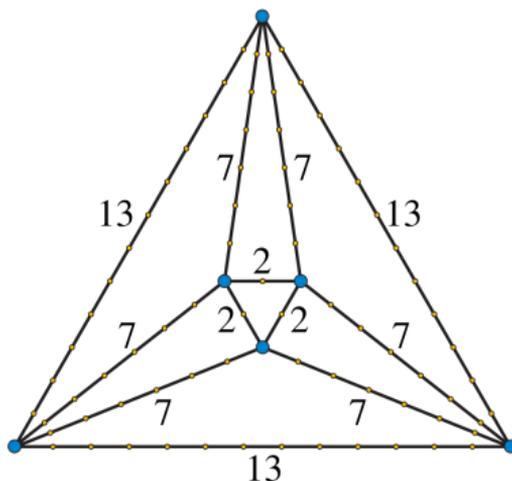**Answer:** Yes! This is known as **Fáry's Theorem** named after István Fáry.

For example...

Even More Questions

Question: Ok so we can always move vertices. But how much do we need to? Can we always move the vertices so that the straight edges are of integer lengths?

Answer: We don't know. Maybe?

We think yes, but nobody has been able to prove it yet. This is **Harborth's Conjecture**.

Even More Slightly Stranger Questions

Ok but we live on a sphere, can we solve Three Utilities if we are allowed to go all the way around the world? Still no :(

Ok but imagine if we lived on a donut, **then** can we solve Three Utilities? **YES!** :)

What about if we lived on a mug?